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ON THE VARIETIES PARAMETRIZING RATIONAL SPACE CURVES WITH FIXED NORMAL BUNDLE.

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Introduction

This note complements, on the one hand, the results obtained in [G.S] for rational curves, on the other hand it gives an introduction to the study of the normal bundle of curves in $\mathbb{P}^3$ (projective space over an algebraically closed field of characteristic 0) of arbitrary genus. In fact, the preliminary results are obtained without hypotheses on the genus of the curves.

Let $C \subset \mathbb{P}^3$ be a rational curve with, at worst, ordinary singularities (i.e., if C is given as the image of a morphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^3$, the map $\varphi^*\Omega^1_{\mathbb{P}^3} \to \Omega^1_{\mathbb{P}^1}$ is surjective), $\mathcal{N}$ the normal bundle of C in $\mathbb{P}^3$. We prove the following Theorem.

THEOREM. Let U^n (resp. T^n) denote the irreducible space of dimension $4n$ parametrizing rational curves in $\mathbb{P}^3$ of degree n with, at worst, ordinary singularities (resp. that are smooth).

There exists a stratification of U^n (resp. T^n)

$$\emptyset \neq U^n_{n-3} \subset U^n_{n-4} \subset \ldots \subset U^n_\rho \subset \ldots \subset U^n_1 \subset U^n$$

$$(\text{resp. } \emptyset \neq T^n_{n-4} \subset T^n_{n-5} \subset \ldots \subset T^n_\rho \subset \ldots \subset T^n_1 \subset T^n)$$

such that

1) $C \in U^n_\rho$ (resp. $C \in T^n_\rho$) $\Longleftrightarrow \mathcal{N} \cong \mathcal{O}_{\mathbb{P}^1}(2n-1-\bar{\rho}) \oplus \mathcal{O}_{\mathbb{P}^1}(2n-1+\bar{\rho})$

   with $\bar{\rho} \geq \rho$.

2) U^n_ρ (resp. T^n_ρ), $1 \leq \rho \leq n-3$ (resp. $1 \leq \rho \leq n-4$)

   is a quasi-projective, integral, locally Cohen-Macaulay

1.

In this section $C \subset \mathbb{P}^3$ denotes a smooth, connected curve of degree n and genus g. (The smoothness assumption is made to simplify notations, it is enough to assume that C has at worst ordinary singularities and consider the normalization $\tilde{C} \to C$ of C.)

Let $\pi : F \to C$ be a geometrically ruled surface over C, i.e., F is of the form $F \approx \mathbb{P}(\mathcal{F})$, where $\mathcal{F}$ is a locally free sheaf of rank 2 on C and π is the canonical projection.

Let H be a unisecant divisor on F, i.e., $(H, F_x) = 1$ for all $x \in C$ and $F_x = \pi^{-1}(x)$, and assume $\mathcal{O}_F(H) = \mathcal{O}_{\mathcal{F}}(1)$. Consider a linear system $\Sigma \subset |H|$, without base points, such that $\dim \Sigma = r \geq 3$ and assume that the corresponding morphism

$$\delta : F \to \mathbb{P}^r$$

is birational onto its image. Then we shall say that $R = \delta(F) \subset \mathbb{P}^r$ is a ruled surface over C. It results that the fibres of F become lines in $\mathbb{P}^r$ and that the degree of R is $q = \deg \mathcal{F}$. Moreover, δ gives a 1-1 correspondence between the sections of $\pi : F \to C$ and the directrices of R. Also, if $D \subset F$ is a section of F, given by the surjection $\mathcal{F} \to \mathcal{L}_D \to 0$, then the degree of the directrix $\delta(D) \subset R$ is given by

$$(1) \qquad \deg \delta(D) = \deg \mathcal{L}_D .$$

It follows that if D_1 and D_2 are two directrices of R, then

$$(2) \qquad (D_1, D_2) = \deg D_1 + \deg D_2 - q ,$$

for $q \in \mathbb{P}^3$.

We have

$$(7) \qquad \deg C^{\vee} = \deg(\mathcal{N}(-1)) = 2n + 2g - 2$$

As an immediate consequence of (2) and (3), we get the following

PROPOSITION 1-1. 1) The surjections $\mathcal{N} \to \mathcal{L} \to 0$ correspond one-to-one to the directrices of degree $\deg \mathcal{L} - n$ of $C^{\vee}$.
2) $\mathcal{N}$ decomposes if and only if there exist two directrices D_1, D_2 such that $\deg D_1 + \deg D_2 = 2n + 2g - 2$.

<u>REMARK 2-2</u>. If $D = \check{C}q$, i.e. if D is a plane directrix of $\check{C}$, then by duality C belongs to the cone $\check{C}q$ with vertex q. Moreover, if $\check{C}q$ projects C birationally, a generic plane section of $\check{C}q$ coincides with the dual plane curve of D (see [P]).

The construction used in Lemma 2-1 gives a geometrical explanation of the fact observed in [G-S] concerning the biduality that exists between the curves of $\mathbb{P}^3$ and the curves of $\check{\mathbb{P}}^3$ considered as sections of $\check{\mathcal{N}}$.

A consequence of Lemma 2-1 is the existence, for each directrix D of C (resp. for each quotient $\mathcal{N} \to \mathcal{L}_D \to 0$), of a "canonical" surface which contains C simply, and such that its tangent planes along C give D (resp. $\mathcal{L}_D$). Clearly this surface is $\check{D}$. Since $\deg \check{D} \leq 2 \deg D + 2g - 2$, we obtain

<u>COROLLARY 2-3</u>. A directrix of $\check{C}$ of degree $\leq d$ comes from a surface of degree $\leq 2d + 2g - 2$.

This formula also gives the degree of the cone $D^{\vee}$ in case D is plane.

From Proposition 3-1, taking into account the formula (8), it follows:

COROLLARY 3-2. Let $D \subset \mathbb{P}^3$ be a curve. The directrices of degree μ on $D^{\vee}$ are parametrized by a smooth, irreducible, quasi-projective variety of dimension $\leq 2\mu - 2d + 1 - K_0(D)$.

The following hold:

1) $C \in V^n_\rho \Leftrightarrow \mathcal{N} \simeq \mathcal{O}(2n-1-\bar{\rho}) \oplus \mathcal{O}(2n-1-\bar{\rho})$, with $\bar{\rho} \geq \rho$.

2) $V^n_\rho \neq \emptyset \Leftrightarrow 0 \leq \rho \leq n-3$

3) $C \in B^n_{n-3} \Leftrightarrow C$ is on a quadric cone and has a point of multiplicity $n-2$ at the vertex.

4) (see Prop. 3-5 [G.S]) Set $d = n-1-\rho$.

Consider the correspondence

$$\Gamma_{n,d} = \{(\psi, f) \in \mathbb{P}^{4n+3} \times \mathbb{P}^{4d+3} : f \in \mathbb{P}(\operatorname{Ker} \omega_{n+d}(\psi))\},$$

where $\Gamma_{n,d}$ as a subvariety of $\mathbb{P}^{4n+3} \times \mathbb{P}^{4d+3}$ is defined by the equations

$$\sum_{i=0}^{3} \frac{\partial \psi_i}{\partial t_j} f_i = 0, \quad j = 0,1. \tag{9}$$

Consider the natural projections p_1 and p_2

$$\begin{tikzcd} & \Gamma_{n,d} \subset \mathbb{P}^{4n+3} \times \mathbb{P}^{4d+3} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{P}^{4n+3} & & \mathbb{P}^{4d+3} \end{tikzcd}$$

a) p_1 is an isomorphism above $V^n_{n-1-d} - V^n_{n-d}$.

b) There exists a unique irreducible component $\Gamma^{(1)}_{n,d}$ of $\Gamma_{n,d}$ such that

$$p_2(\Gamma^{(1)}_{n,d}) = \mathbb{P}^{4d+3}.$$

Set $A_\rho = A \cap (V^n_\rho - V^n_{\rho+1})$. Clearly

$$p(A\rho) \subset (\overline{V}_1^{n-1-\rho} \cap W^{n-1-\rho})$$

(see (4,d)).

Let us prove that $\dim A \leq 4n+3-2\rho$. If $f \in V_1^{n-1-\rho}$, denote by D_f the curve defined by f, then $k_o(D_f) = 0$. Taking into account Cor. 3-2, we get $\dim p^{-1}(f) = 2\rho+5$ and hence $\dim p^{-1}(V_1^{n-1-\rho}) \leq 4n+3-2\rho$. Now if $f \in (\overline{V}_1^{n-1-\rho} \cap W^{n-1-\rho}) - V_1^{n-1-\rho}$, then $K_o(D_f) \neq 0$.

Set

$$W_h^{n-1-\rho} = \{f \in W^{n-1-\rho} : K_o(D_f) \geq h\}, \quad h \geq 1 .$$

There is a stratification of $W^{n-1-\rho}$

$$\ldots \subset W_h^{n-1-\rho} \subset W_{h-1}^{n-1-\rho} \subset \ldots \subset W_1^{n-1-\rho} \subset W^{n-1-\rho} .$$

Now $\operatorname{codim}_{W^{n-1-\rho}} W_h^{n-1-\rho} \geq 2h$ (as can be easily seen for example by considering rational curves of degree n in $\mathbb{P}^3$ as projections of normal curves in $\mathbb{P}^n$), whereas the dimension of the fibre of p on $W_{h-1}^{n-1-\rho} - W_h^{n-1-\rho}$ increases only by h (Cor.3-2). This shows that $\dim A \leq 4n+3-2\rho$.

We thus conclude, using the Observation 4-1. This also proves that V^n_ρ is locally Cohen-Macaulay, because it is a determinantal variety of maximal codimension. Moreover, it follows that $\Gamma_{n,n-1-\rho} = \Gamma^{(1)}_{n,n-1-\rho}$ and hence is a complete intersection (defined by the $2(n+d) =$
$= \operatorname{codim}_{\mathbb{P}^{4n+3} \times \mathbb{P}^{4(n-1-\rho)+3}} \Gamma_{n,n-1-\rho}$ equations obtained by (9)).

5.

As we have already seen in the preceding section (3), the curves of V^n_{n-3} have a precise geometrical characterization. Therefore it is natural to ask whether this is so in general. The answer is positive in the case $\rho = n-4$.

PROPOSITION 5-1. Let C be smooth, of degree $n \geq 6$ $(C \in T^n)$. Then:

$$C \in T^n_{n-4} \iff \begin{array}{l} C \text{ is a directrix of the tangent} \\ \text{developable of a twisted cubic.} \end{array}$$

PROOF. Assume $\mathcal{N} \cong \mathcal{O}(n+3) \oplus \mathcal{O}(3n-5)$, then there exists (Prop. 1-1) a directrix (clearly irreducible and simple) $D \subset C^\vee$ such that $\deg D = 3$. The curve D cannot be plane, otherwise, since $n \geq 6$, C would have a singular point at the vertex of the cone $D^\vee$. We conclude by applying Lemma 2-1, since twisted cubics are self-dual.

The converse follows immediately from similar considerations.

If $\rho \leq n-5$, the geometrical situation is not so clear. In fact, for a rational curve D of degree ≥ 4 there are numerous possibilities for $\deg D^\vee$ and $\deg D^*$.

It seems that the best approach to the classification problem would be to consider the curves of $\mathbb{P}^3$ as projections of normal curves in $\mathbb{P}^n$.

REFERENCES

[G-S] F. GHIONE, G. SACCHIERO: Normal Bundles of rational curves. Manuscripta Math. 33, 111-128 (1980)

[H] R. HARTSHORNE: Algebraic Geometry. Springer-Verlag (1977)

[P] R. PIENE: Numerical characters of a curve in projective n-space. Real and complex singularities, Oslo 1976. Sijthoff en Noordhoff. 475-495.

[S] C. SEGRE: Sulle rigate razionoli in uno spazio lineare qualunque. Atti delle R. Acc. delle Scienze di Torino **XIX**, 265-282 (1883-84)